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Discrete analytic functions

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DISCRETE ANALYTIC FUNCTIONS

by

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CONTENTS

Abstract

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The theory and application of discrete harmonic and discrete analytic functions have received considerable attention in mathematical literature. One definition of discrete analyticity was introduced by Jacqueline Ferrand (Lelong). She developed several interesting analogies with ordinary analytic functions. Rufus Isaacs contrived a theory based on another definition of analyticity. R. J. Duffin extended their work in several directions. Duffin's new developments include analogies of the Cauchy Integral Formula and the Maximum Modulus Principle.

The work of Isaacs and Duffin shows that it is difficult to define a product for two discrete analytic functions, which preserves analyticity. This problem was solved by C. S. Duris using a "convolution product."

This thesis is a report of the work of Duffin and Duris, which have appeared only in research publications. It contains both a detailed explanation of their work and the proofs of many results merely stated by them.

Introduction

This thesis is concerned with complex-valued functions defined at the points of the complex plane whose coordinates are integers. These points form a lattice which breaks up the plane into unit squares. A function is said to be discrete analytic at one of these squares if the difference quotient across one diagonal is equal to the difference quotient across the other diagonal; i.e.

$$\text{(c)} \qquad [f(z + 1 + i) - f(z)]/(1 + i) = [f(z + i) - f(z + 1)]/(i - 1).$$

If $f = u + iv$ where u and v are real, then it follows that discrete analyticity implies that u and v satisfy a pair of difference equations which are analogous to the Cauchy-Riemann equations. If f is discrete analytic in a region it follows that u and v are discrete harmonic in that region. That is, they satisfy the Laplacian difference equations.

The above definition of analyticity was introduced by Jacqueline Ferrand (Lelong) [3]. She developed several interesting analogies with ordinary analytic functions. R. J. Duffin [1] extended her work in several directions. These new developments include analogies of the Cauchy integral formula and the maximum modulus principle. Some, however, have no direct analogy in the classical continuous theory. Among these is the notion of duality.

The theory and the application of discrete harmonic functions have received considerable attention in the literature. Much of this work may be brought to bear on the present problem.

Rufus Isaacs [4,5] developed a theory of discrete analytic functions based on the following definition of analyticity:

(a) $f(z + 1) - f(z) = [f(z + i) - f(z)]/i.$

He preferred this definition to

(b) $f(z + 1) - f(z - 1) = [f(z + i) - f(z - i)]/i$

which he also considered. It appears that definition (b) is essentially equivalent to the Ferrand definition (c). It is not apparent that (a) and (c) have any direct relationship. Definition (a) is simpler than (c) and leads to simpler algebraic formulae. On the other hand the harmonic functions corresponding to (a) do not satisfy the standard Laplacian difference equation.

The work of Isaacs and Duffin shows that it is difficult to define a product for two discrete analytic functions which preserves discrete analyticity. They investigate operators which correspond to multiplication of continuous analytic functions, but these operators preserve analyticity only if one of the factors is a polynomial. Here, a different approach, due to C. S. Duris [2], is taken to the problem of multiplication. Using the definition of a discrete line integral given by Duffin [1], a "convolution product" of two discrete functions is defined, which is commutative, associative and distributive; and if the functions involved are discrete analytic, then so is the product. This product is a summation over a chain of lattice points, which resembles the convolution integral appearing in the theory of the Laplace transform.

Synonyms for "discrete analytic" are "preholomorphic" and "monodiffric." "Preharmonic" is synonymous with "discrete harmonic." Various other discrete analogs of classical continuous concepts are to be introduced; when no confusion results, the qualifier "discrete" is dropped.

1. Conjugates, Duals, and Derivatives[1]

Definitions and Remarks

DEFINITION 1.1. The lattice points of the complex plane are the points $z = m + in$ where m and n take on the values 0, ± 1, ± 2, ...

Of concern in this paper are complex-valued functions defined at the lattice points.

DEFINITION 1.2. If z_o is a lattice point, the points z_o, $z_o + 1$, $z_o + 1 + i$, $z_o + i$ are the vertices of a unit square associated with the point z_o.

A unit square is regarded as a closed two dimensional point set.

DEFINITION 1.3. A region is the union of unit squares.

DEFINITION 1.4. A simple region is a simply connected set which is the union of a finite number of unit squares.

Thus the boundary of a simple region is a simple closed curve which is composed of edges of unit squares.

For example, Figure 1 depicts the union of seven unit squares which form a simple region R.

[1]The definitions and results appearing in this chapter may be found in [1].

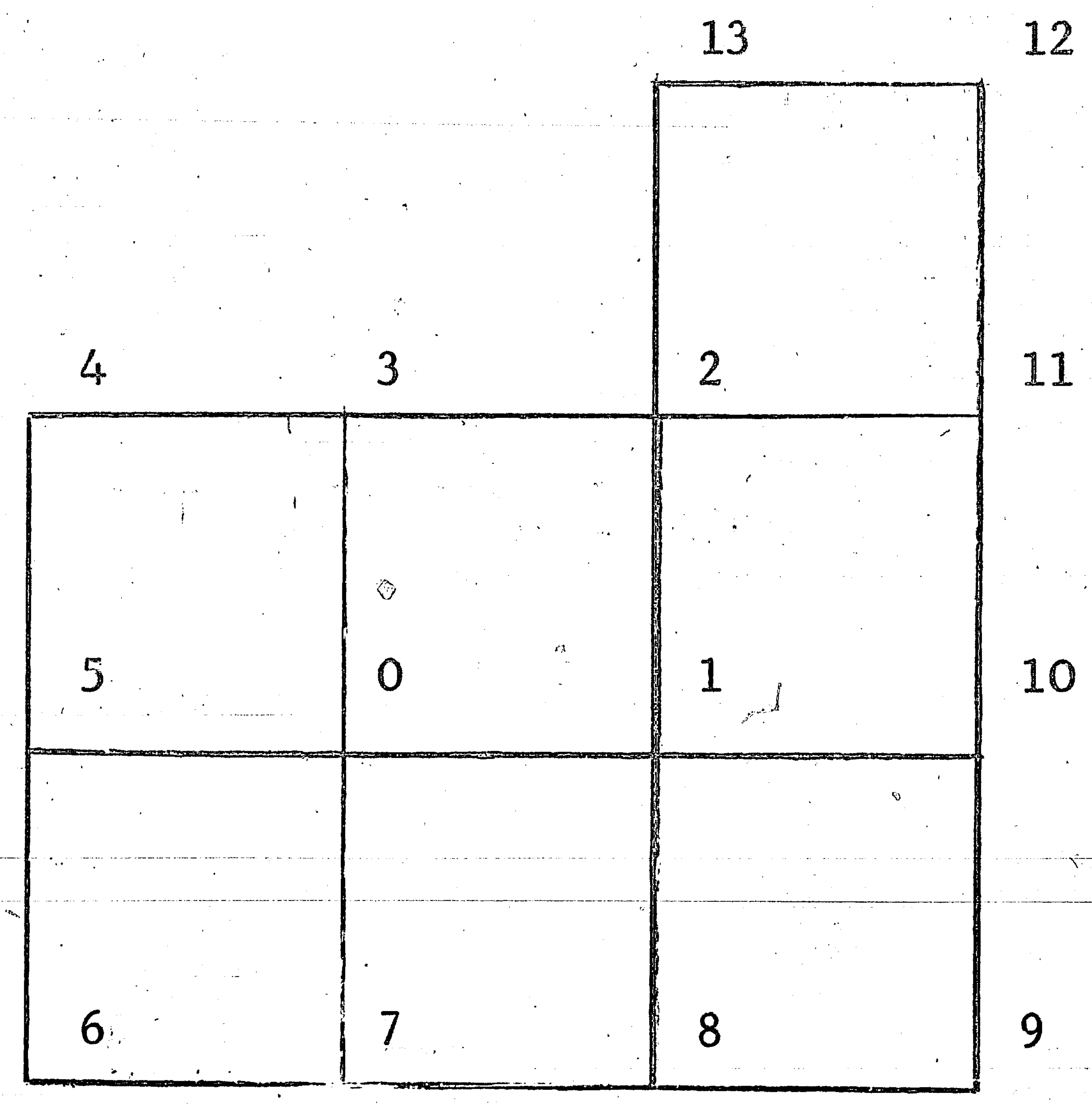

Figure 1

The lattice points of R are z_o, $z_1 = z_o + 1$, $z_2 = z_o + 1 + i, \ldots, z_{13} = z_o + 1 + 2i$. If f is defined at these points then as a short notation let $f_n = f(z_n)$.

<u>DEFINITION</u> 1.5. The <u>even</u> (odd) <u>lattice</u> consists of those points for which $n + m$ is even (odd), where $z = m + in$.

Thus if z_o is on the even lattice, so also are $z_2, z_4, \ldots, z_{12}$ while $z_1, z_3, \ldots, z_{13}$ are on the odd lattice.

<u>DEFINITION</u> 1.6. The lattice points of a region which are not on the boundary are termed <u>interior points</u>.

Thus in the example z_o and z_1 are interior points; the other points are boundary points. The scheme indicated in Figure 1 gives a method of listing the lattice points of the plane as a single sequence $z_o, z_1, z_2, \ldots$ This sequence may be termed the <u>spiral coordinate system</u>.

<u>DEFINITION</u> 1.7. A function f is said to be <u>discrete analytic</u> on the square associated with z_o if

(1) $$f_o + i\, f_1 + i^2 f_2 + i^3 f_3 = 0.$$

To see that this definition is consistant with definition (c) given in the introduction we note that:

(c) $$\frac{f_2 - f_o}{i + 1} = \frac{f_3 - f_1}{i - 1}.$$

This implies that

$$(i - 1)(f_2 - f_o) = (i + 1)(f_3 - f_1).$$

If we multiply both sides by $\frac{i}{1 + i}$ we obtain:

$$\frac{i(i - 1)}{1 + i}[f_2 - f_o] = \frac{i(i + 1)}{1 + i}[f_3 - f_1],$$

or

$$i^2[f_2 - f_o] = i[f_3 - f_1],$$

or

$$-i^2 f_o + if_1 + i^2 f_2 - if_3 = 0,$$

which is the same as Definition 1.7,

$$f_o + if_1 + i^2 f_2 + i^3 f_3 = 0 \, .$$

To handle such relations it is convenient to introduce translation operators X and Y defined by

$$(2) \qquad X^n f(z) = f(z + n) \, , \; Y^n f(z) = f(z + in)$$

$$\text{for } n = 0, \pm 1, \pm 2, \ldots$$

<u>LEMMA</u> 1.1. <u>If</u> $X^n f(z) = f(z + n)$ <u>and</u> $Y^n f(z) = f(z + in)$ <u>for</u> $n = 0, \pm 1, \pm 2, \ldots$ <u>then</u>

$$\text{(i)} \quad X^n X^m = X^{m+n}$$

$$\text{(ii)} \quad X^n Y^m = Y^m X^n$$

$$\text{(iii)} \quad Y^n Y^m = Y^{n+m} \, .$$

PROOF: (i)

$$\begin{aligned} [X^n X^m] \, f(z) &= X^n [X^m (f(z))] \\ &= X^n f(z + m) \\ &= f((z + m) + n) \\ &= f(z + (m + n)) \\ &= X^{m+n} f(z). \end{aligned}$$

(ii)

$$\begin{aligned} [X^n Y^m] \, f(z) &= X^n [Y^m f(z)] \\ &= X^n f(z + im) \\ &= f((z + im) + n) \\ &= f(z + (im + n)) \end{aligned}$$

$$
\begin{aligned}
[X^nY^m]\ f(z) &= f(z + (n + im)) \\
&= f((z + n) + im) \\
&= Y^m\ f(z + n) \\
&= Y^m\ [X^n\ f(z)] \\
&= [Y^mX^n]\ f(z)
\end{aligned}
$$

(iii) This follows clearly from the proof of (i) by noting that $Y^n = X^{in}$, $n = 0, \pm 1, \pm 2, \ldots$

Q.E.D.

LEMMA 1.2. *Let*

(3) $$L = I + iX - YX - iY,$$

where I *is the identity operator, then the analyticity condition may be expressed as* Lf = 0.

PROOF:

$$
\begin{aligned}
Lf(z_o) &= If(z_o) + iXf(z_o) - YXf(z_o) - iYf(z_o) \\
&= f(z_o) + if(z_o+1) - Yf(z_o+1) - if(z_o+i) \\
&= f(z_o) + if(z_o+1) - f(z_o+1+i) - if(z_o+i) \\
&= f(z_o) + if(z_1) - f(z_2) - if(z_3)
\end{aligned}
$$

(4) $$= f_o + if_1 + i^2f_2 + i^3f_3.$$

Q.E.D.

THEOREM 1.1. *Let* f = u + iv *where* u *and* v *are the real and imaginary parts of* f. *Then* (1) *is equivalent to*

(5) $\quad u_2 - u_o = v_3 - v_1 \quad \text{and} \quad u_3 - u_1 = v_o - v_2 .$

PROOF: $\quad f_o + if_1 + i^2f_2 + i^3f_3 = 0$

if and only if

$$(u_o+iv_o)+i(u_1+iv_1)+i^2(u_2+iv_2)+i^3(u_3+iv_3) = 0$$

if and only if

$$u_o+iv_o+iu_1-v_1-u_2-iv_2-iu_3+v_3 = 0$$

if and only if

$$u_o-v_1-u_2+v_3 = 0 \quad \text{and} \quad v_o+u_1-v_2-u_3 = 0$$

if and only if

$$u_2-u_1 = v_3-v_1 \quad \text{and} \quad v_o-v_2 = u_3-u_1 .$$

Q.E.D.

These are the discrete analogs of the Cauchy-Riemann equations.

DEFINITION 1.8. A function is said to be *analytic in a region* if it is analytic in each unit square of the region.

LEMMA 1.3. *The group of rigid motions which transform the lattice into itself may be generated by the translations X and Y, a 90° rotation, and a reflection in the x-axis. These motions preserve analyticity.*

PROOF: Of course analyticity is preserved under translation. Given a function f then a 90° rotation defines a function $F(z) = f(-iz)$. If

$Lf = \omega$ at a certain square then at the rotated square

$LF = f_3 + if_o - f_1 - if_2 = i\omega$.

Thus 90^o rotations preserve analyticity. To consider reflections let $F(z) = f^*(z^*)$ where the star denotes the complex conjugate. At the reflected square $LF = f_3^* + if_2^* - f_1^* - if_o^*$ $= -i\omega^*$. Thus reflection plus conjugation preserves analyticity and the lemma is proved.

Q.E.D.

DEFINITION 1.9. A function which is discrete analytic at every lattice point is called an entire function.

THEOREM 1.2. *If a function f is prescribed arbitrarily at the lattice points of the x and y axis, then f has a unique continuation as an entire function.*

PROOF: To see this, attention is focused on the first quadrant. Then $Lf = 0$ evaluates $f(i + 1)$, $f(i + 2)$, $f(i + 3)$,... in succession. Then $f(2i + 1)$, $f(2i + 2)$,... are evaluated in succession. In this manner a continuation of f is determined in the first quadrant. A similar procedure applies to the other quadrants. This continuation is unique by construction.

Q.E.D.

The following device is useful to simplify the process of continuation. Given a function f, let an associated function g be defined by

$$g(z) = i^{x-y}f(z), \text{ where } z = x + iy.$$

Then

$$i^{x-y}Lf = i^{x-y}[f_o + if_1 + i^2f_2 + i^3f_3]$$

$$= i^{x-y}f_o + i^{x+1-y}\, f_1 + i^{x-y+2}\, f_2 + i^{x-y+3}\, f_3$$

$$= g_o + g_1 - g_2 + g_3 \,.$$

Thus f(x,y) is analytic if and only if

(6) $$g(x+1,y+1) = g(x,y) + g(x+1,y) + g(x,y+1).$$

In particular let ψ satisfy this relation and the conditions $\psi(0,y) = 1$ for all y and $\psi(x,0) = 0$ for all x except $x = 0$. Then it is clear $\psi(x,y) = 0$ for $x < 0$. Then function ψ is essentially a two dimensional Fibonacci series of integers.

THEOREM 1.3. Let $\psi(z)$ be the entire function associated with $\psi(z)$. Let f(z) be an arbitrary entire function, then

(7) $$f(z) = f(0) - f(0)\sum_{k=1}^{\infty}\Big\{\psi(z-k) + \psi(-z-k) + \psi(-iz-k) + \psi(iz-k)\Big\} + \sum_{k=1}^{\infty}\Big\{f(k)\,\psi(z-k) + f(-k)\,\psi(-z-k) + f(ik)\,\psi(-iz-k) + f(-ik)\,\psi(iz-k)\Big\}.$$

PROOF: Note that the series converges since for any given z there are only a finite number of non-vanishing terms. It is clear that the expression on the right is an entire function. However the left and right sides agree on the real and imaginary axes. Thus, by Theorem 1.2, this completes the proof of the expansion theorem.

Q.E.D.

<u>DEFINITION</u> 1.10. Two functions f and F are said to be <u>dual</u> if $F = (-1)^{x+y}f^*$ where the star denotes the complex conjugate.

<u>LEMMA</u> 1.4. <u>*A function and its dual have the same region of analyticity.*</u>

PROOF: Let $z = x + iy$ then

$$(Lf)^* = (f_o + if_1 + i^2f_2 + i^3f_3)^*$$

$$= (f_o^* - if_1^* - f_2^* + if_3^*)$$

$$= (-1)^{x+y} F_o - i(-1)^{x+y+1} F_1 - (-1)^{x+y}F_2 + i(-1)^{x+y+1} F_3$$

$$= (-1)^{x+y}[F_o + iF_1 - F_2 - iF_3]$$

$$= (-1)^{x+y} LF.$$

So $Lf = 0$ if and only if $LF = 0$.

Q.E.D.

The dual of a function f will be denoted by f^{-}.

Let

(8) $$L' = I - iX^{-1} - Y^{-1}X^{-1} + iY^{-1}.$$

Then

$$L'f(z_o) = f_o - if_5 - f_6 - if_7$$

and

$$(I - iX^{-1} - Y^{-1}X^{-1} + iY^{-1})(I + iY - YX - iY)$$

$$= I + iX - YX - iY - iX^{-1} - iX^{-1}iX + iX^{-1}YX$$

$$+ i^2X^{-1}Y - Y^{-1}X^{-1} - iY^{-1}X^{-1}X + Y^{-1}X^{-1}YX$$

$$+ iY^{-1}X^{-1}Y + iY^{-1} + i^2Y^{-1}X - iYXY^{-1} - i^2Y^{-1}Y$$

$$= I + iX - YX - iY - iX^{-1} + I + iY - X^{-1}Y$$

$$- Y^{-1}X^{-1} - iY^{-1} + I + iX^{-1} + iY^{-1} - Y^{-1}X - iX + I$$

$$= 4I - YX - X^{-1}Y - Y^{-1}X^{-1} - Y^{-1}X.$$

This last relation may be written as

(9) $$L'L = LL' = -D$$

where $-Df(z_o) = 4f_o - f_2 - f_4 - f_6 - f_8$ or

(10) $$Df(z_o) = f_2 + f_4 + f_6 + f_8 - 4f_o.$$

DEFINITION 1.11. A function is said to be harmonic at a point z_o if $Df(z_o) = 0$.

Remarks

It is seen from (10) that if f is analytic in a region R, then f is harmonic at the interior points of R. The Laplacian operator D is real, so the real and imaginary parts of f are also harmonic. It is to be noted that the operator D does not interrelate functional values on the even lattice with functional values on the odd lattice.

The usual definition of the Laplacian difference operator is not (11) but

$$D_o g(x,y) = g(x+1,y) + g(x-1,y) + g(x,y+1) + g(x,y-1) - 4g(x,y).$$

To relate D_o to D let A be the transformation $x' = x +, y$, $y' = x - y$. Then A is a rotation of 45^o followed by a dilation of value $\sqrt{2}$. It is clear that A gives a one-to-one mapping of the whole lattice onto the even lattice. Then $A(x \pm 1, y) = (x' \pm 1,\ y' \pm 1)$ and $A(x, y \pm 1) = (x' \pm 1,\ y' \mp 1)$. Let $f(x', y') = g(x,y)$ then clearly

$$Df(x',y') = D_o g(x,y).$$

This relation shows that any function harmonic with respect to D_o may be transformed into a function harmonic with respect to D simply by a change of coordinates.

This transformation will be tacitly employed in the parts to follow.

Given a lattice function h, let f be defined by

$$f = L'h. \tag{11}$$

Then $Lf = LL'h = -Dh$. Thus if h is harmonic at a point z_o, f is analytic on the square associated with z_o.

THEOREM 1.4. Let u be a real function defined on a simple region R and harmonic at the interior points of R. Then u is the real part of a discrete analytic function f which is discrete analytic in R.

PROOF: This may be proved by induction. The theorem is obviously true if R is a unit square. Suppose that it is true for all regions containing fewer than n squares, and consider a region R with n squares. By geometric intuition one sees that it is possible to delete one of the squares of R, say S, to leave a simple region R'. One of the edges of S must be on the boundary of R; otherwise R' would be doubly connected. Moreover, one of the edges of S is on the boundary, but the opposite edge is not on the boundary; otherwise R' would be disconnected. Referring to Figure

1, it may be supposed that S is the square whose lower left corner is at z_o. Suppose also that the edge (2,3) is on the boundary of R but that (0,1) is not on the boundary. This is no loss of generality, because any other case may be reduced to this by a rotation.

By assumption a conjugate function may be assigned to the lattice points of R'. If the point z_3 is in R', then the square at z_5 is in R, for otherwise R would not be bounded by a simple closed curve. Also, the square at z_6 is in R, for otherwise R' would not be bounded by a simple closed curve. It follows that z_o is an interior point of R, so Du = 0 at z_o. Thus

$$(u_2 - u_o) + (u_4 - u_o) + (u_6 - u_o) + (u_8 - u_o) = 0.$$

Making use of the Cauchy-Riemann analogs for the squares at z_5, z_6, and z_7 gives

$$(u_2 - u_o) + (v_5 - v_3) + (v_7 - v_5) + (v_1 - v_7) = 0.$$

Thus $u_2 - u_o = v_3 - v_1$, and the first Cauchy-Riemann equation is satisfied. If z_3 is not in R', the value of v_3 may be assigned to satisfy this equation. A symmetrical argument applies to the point z_2 and the second Cauchy-Riemann equation. The function v so extended is conjugate to u in S and hence in all of R. Let $f = u + iv$, then f is discrete analytic in R and the proof is completed.

Q.E.D.

<u>THEOREM</u> 1.5. Maximum Principle

<u>If f is harmonic in a finite region, R, then |f| takes on its maximum on the boundary of that region.</u>

PROOF: If f is a constant function the theorem is trivially true; so suppose f is non-constant. Let S be the set of boundary points of R, and $T = \left\{ |f(z)| : z \in R-S \right\}$. Since R contains only a finite number of points, there exists a point $z_o \in R-S$ such that $|f(z_o)| = \max \left\{ |f(z)| : |f(z)| \in T \right\}$. If there is more than one point z_o available, pick that one such that $z_o + 1 + i$, $z_o + 1 - i$, $z_o - 1 - i$, and $z_o - 1 + i$ are not all equal. If this is impossible the theorem follows trivially.

Since f is harmonic at z_o

(12) $$f_2 + f_4 + f_6 + f_8 = 4f_o,$$

where by definition, $f_2 = f_o + 1 + i$, $f_4 = f_o + i - 1$, $f_6 = f_o - i - 1$, $f_8 = f_o + 1 - i$. Let $|f_m| = \mathrm{Max} \left\{ |f_2|, |f_4|, |f_6|, |f_8| \right\}$.

$$4|f_o| = |f_2 + f_4 + f_6 + f_8|$$
$$\leq |f_2| + |f_4| + |f_6| + |f_8|$$
$$< 4|f_m|.$$

Thus $|f_m| > |f_o|$. By the choice of z_o, z_m must be on the boundary of R.

Q.E.D.

Now let $z_o, z_1, z_2, \ldots, z_m$ denote any chain of lattice points, that is $|z_o - z_1| = 1$, $|z_1 - z_2| = 1$, etc. (This sequence is not necessarily the spiral coordinate system.) Let $a = z_o$ and $b = z_m$.

<u>DEFINITION</u> 1.12. C is a <u>closed</u> <u>chain</u> if a = b.

<u>DEFINITION</u> 1.13. The "<u>line</u> <u>integral</u>," $\int_a^b f\,\delta z$, is defined by

$$\int_a^b f\,\delta z = \frac{1}{2}\sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1}). \tag{13}$$

It is obvious from this definition that

$$\int_a^c f\,\delta z = \int_a^b f\,\delta z + \int_b^c f\,\delta z, \tag{13a}$$

$$\int_a^b kf\,\delta z = k\int_a^b f\,\delta z \quad \text{where } k \text{ is a constant,} \tag{13b}$$

$$\int_a^b (f + g)\delta z = \int_a^b f\,\delta z + \int_a^b g\,\delta z, \tag{13c}$$

(13d) $$\int_a^b f \, \delta z = -\int_b^a f \, \delta z.$$

LEMMA 1.5. *If C is a closed chain*

$$\int_C f \, \delta z = \frac{1}{2}\sum_{n=1}^{m} f_n(z_{n+1} - z_{n-1}).$$

PROOF: $$\frac{1}{2}\sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1})$$

$$= \frac{1}{2}\sum_{n=1}^{m} [f_n z_n + f_{n-1} z_n - f_n z_{n-1} - f_{n-1} z_{n-1}]$$

$$= \frac{1}{2}\sum_{n=1}^{m} [f_{n-1} z_n - f_n z_{n-1}] \quad \text{since } z_o = z_m$$

$$= \frac{1}{2}\sum_{n=1}^{m} [f_{n-1} z_n] - \sum_{n=1}^{m} [f_n z_{n-1}]$$

$$= \frac{1}{2}\sum_{n=1}^{m} f_{n-1} z_n - \sum_{n=2}^{m+1} [f_{n-1} z_{n-2}]$$

$$= \frac{1}{2}\left\{f_o z_1 + \sum_{n=2}^{m} [f_{n-1}(z_n - z_{n-2})] - f_m z_{m-1}\right\}$$

$$= \frac{1}{2}\left\{f_o z_1 + \sum_{n=1}^{m-1} [f_n(z_{n+1} - z_{n-1})] - f_m z_{m-1}\right\}$$

$$= \frac{1}{2}\left\{\sum_{n=1}^{m-1} [f_n(z_{n+1} - z_{n-1})] + f_m z_{m+1} - f_m z_{m-1}\right\}$$

where $z_1 = z_{m+1}$.

Thus $\frac{1}{2}\sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1})$

$$= \frac{1}{2}\sum_{n=1}^{m} f_n(z_{n+1} - z_{n-1}).$$

Q.E.D.

LEMMA 1.6. Let the contour (chain) be taken in a counter clockwise direction around a unit square. Then

$$2\int_C f\ \delta z = (1 - i)\ Lf.$$

PROOF: By Lemma 1.5

$$2\int_C f\ \delta z = f_o(z_1 - z_3) + f_1(z_2 - z_o)$$

$$+ f_2(z_3 - z_1) + f_3(z_o - z_2).$$

However, $z_1 = z_3 + 1 - i$, $z_2 = z_o + 1 + i$, $z_3 = z_1 + i - 1$. Thus

$$2\int_C f\ \delta z = (1 - i)f_o + (1 - i)f_1$$

$$+ (-1 + i)f_2 + (-1 - i)f_3$$

$$= (1 - i)\ [f_o + if_1 - f_2 - if_3]$$

$$= (1 - i)\ Lf.$$

Q.E.D.

It is clear that the line integral around a simple region is a sum of the line integrals around the unit squares. Thus if B is the boundary of a simple region R, then

$$\int_B f\ \delta z = \frac{(1 - i)}{2} \sum_R Lf \tag{14}$$

where $\sum_R$ denotes a summation over the unit squares contained in R.

<u>THEOREM</u> 1.6. $-(1 + i)\ Lf/4\pi$ <u>corresponds to the residue in the classical continuous case</u>.

PROOF: If relation (14) is compared with the classical continuous case, then it is seen that

$$2\pi i \sum \text{Res} = \frac{1 - i}{2} \sum Lf$$

or

$$\sum \text{Res} = \frac{-i(1 - i)}{4\pi} \sum Lf$$

$$= \frac{-1 - i}{4\pi} \sum Lf\ .$$

Q.E.D.

There will be no confusion if Lf at a given unit square is termed the <u>residue</u> of f at that square.

<u>THEOREM</u> 1.7. <u>If f is analytic in a simple region</u> R <u>with boundary</u> B, <u>then</u>

$$\int_B f\ \delta z = 0\ .$$

PROOF: By hypothesis $Lf = 0$ at each unit square of R. Therefore from relation (14)

$$\int_B f\,\delta z = \frac{(1 - i)}{2} \sum_R (0) = 0.$$

Q.E.D.

<u>COROLLARY</u> 1.7.1. <u>Let</u> a <u>and</u> z <u>be points of</u> R <u>and let</u>

$$F(z) = \int_a^z f\,\delta z + C \tag{15}$$

<u>where</u> C <u>is a constant</u>. <u>Then</u> F(z) <u>does not depend on the contour joining</u> a <u>and</u> z <u>provided the contour is in</u> R.

PROOF: Let Δ and Γ be two distinct contours joining a and z, and let

$$\int_\Delta f\,\delta z = A \quad \text{and} \quad \int_\Gamma f\,\delta z = B\,.$$

Then $$\int_{\Delta-\Gamma} f\,\delta z = A - B$$

where $-\Gamma$ denotes Γ taken in the reverse direction. However by Theorem 1.7 $A - B = 0$. Hence $A = B$.

Q.E.D.

<u>THEOREM</u> 1.8. <u>If</u> f <u>is analytic in a simple region</u> R, <u>then the line integral</u>

$$F(z) = \int_a^z f\,\delta z + C$$

<u>is a single valued analytic function in</u> R.

PROOF: If p and q are neighboring points

$$F(p) = \int_q^p f\,\delta z + C \qquad \text{by (15),}$$

$$= \frac{1}{2}\sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1}) + C \quad \text{by (13).}$$

$$F(q) = \int_a^q f\,\delta z \qquad \text{by (15),}$$

$$= \int_a^p f\,\delta z + \int_p^q f\,\delta z + C \quad \text{by Corollary 1.7.1.}$$

$$= \frac{1}{2}\sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1})$$

$$+ \frac{[f(q) + f(p)]}{2}(q - p) + C. \qquad \text{by (13).}$$

Thus $\dfrac{F(p) - F(q)}{q - p} = -\dfrac{1}{2}[f(q) + f(p)]$

or

(16) $$\frac{F(p) - F(q)}{p - q} = \frac{f(p) + f(q)}{2}.$$

In particular, for a unit square:

(i) $2(F_2 - F_1) = (z_2 - z_1)(f_2 + f_1)$

(ii) $2(F_1 - F_o) = (z_1 - z_o)(f_1 + f_o)$

By adding i and ii we obtain

$$2(F_2 - F_o) = i(f_2 + f_1) + 1(f_1 + f_o)$$

(iii) $$= f_o + f_1 + if_1 + if_2 .$$

Similarly,

$$2(F_3 - F_1) = -(f_3 + f_2) + i(f_2 + f_1),$$

or

$$\text{(iv)} \quad 2i(F_3 - F_1) = -f_1 - f_2 - if_2 - if_3.$$

Adding iii and iv we obtain,

$$2F_2 - 2F_o + 2iF_3 - 2iF_1 = f_o + f_1 + if_1 + if_2 - f_1 - f_2 - if_2 - if_3$$

$$-2[F_o + iF_1 - F_2 - iF_3] = f_o + if_1 - f_2 - if_3$$

$$-2LF = Lf.$$

Hence LF = 0 for each square of R. It is thereby shown that if f is discrete analytic in a simple region R, the line integral F is a single valued analytic function in R.

Q.E.D.

THEOREM 1.9. Let F(z) be a given discrete analytic function in a simple region R. Let a and b be points of R and let k be an arbitrary constant. Then

$$(17) \qquad f(z) = \left(4\int_b^z F^- \,\delta z + k\right)^-$$

is analytic in R and

$$(17a) \qquad F(z) = \int_a^z f(z)\,\delta z + F(a).$$

The integration paths are assumed to be in R.

PROOF: If we take the dual of (17) we obtain

$$f^{-} = 4\int_{b}^{z} F^{-}\,\delta z + k.$$

Then by (16) if p and q are neighboring points of R

$$f^{-}(p) - f^{-}(q) = 2(p - q)(F^{-}(p) + F^{-}(q)).$$

Take the conjugate of both sides. Note that $(p - q)^{*} = (p - q)^{-1}$ since $p - q = \pm 1$ or $\pm i$. Then

$$(f(p) + f(q))(p - q) = 2(F(p) + F(q)).$$

This is relation (16), and (17a) is a consequence, so the proof is completed.

Q.E.D.

<u>DEFINITION</u> 1.14. The function f defined by (17) is termed the <u>derivative</u> of F.

If k^{-} is termed a <u>biconstant</u>, the derivative is unique up to an arbitrary biconstant.

2. Contour Integrals and Cauchy's Formula[1]

We will now define more general types of line integrals. These integrals also have the important property of vanishing around a closed contour if the functions concerned are analytic. To this end it is convenient to define other difference operators on the unit square. Let the operator φ be defined as

$$\varphi = I + \epsilon X + \epsilon^2 XY + \epsilon^3 Y. \tag{18}$$

Specializing ϵ to be i, -1, -i and 1 defines four specific operators L, T, M, and S given by:

$$(19)\qquad \begin{aligned} Lf &= f_o + if_1 - f_2 - if_3 , \\ Tf &= f_o - f_1 + f_2 - f_3 , \\ Mf &= f_o - if_1 - f_2 - if_3 , \\ Sf &= f_o + f_1 + f_2 + f_3 , \end{aligned}$$

respectively.

If f and g are lattice functions and $a = z_o, z_1, \ldots, z_m = b$ denotes a chain of lattice points then three types of "line integrals" are defined by:

[1]The definitions and results appearing in this chapter may be found in [1].

$$(20)\quad \int_a^b f : g\ \delta z = \sum_{n=1}^{m} (f_n + f_{n-1})(g_n + g_{n-1})(z_n - z_{n-1})/4$$

$$(21)\quad \int_a^b f : g'\delta z = \sum_{n=1}^{m} (f_n + f_{n-1})(g_n - g_{n-1})/2$$

$$(22)\quad \int_a^b f' : g'\delta z = \sum_{n=1}^{m} (f_n + f_{n-1})(g_n - g_{n-1})/(z_n - z_{n-1})$$

Note that the notations on the left are merely symbolic for the precise definitions on the right.

For a closed path B, a = b and the above relation may be reformed as follows:

$$\int_B f{:}g\ \delta z = \frac{1}{4}\sum_{n=1}^{m} (f_n + f_{n-1})(g_n + g_{n-1})(z_n - z_{n-1})/4.$$

From the facts that $0 = m$ and $1 = m + 1$, we obtain

$$\int f{:}g\ \delta z = \frac{1}{4}\sum_{n=1}^{m} [f_n g_n z_n - f_n g_n z_{n-1} + f_{n-1} g_n z_n - f_{n-1} g_n z_{n-1}$$

$$+ f_n g_{n-1} z_n - f_n g_{n-1} z_{n-1} + f_{n-1} g_{n-1} z_n$$

$$- f_{n-1} g_{n-1} z_{n-1}]$$

$$= \frac{1}{4}\left\{\sum_{n=1}^{m} - f_n g_n z_{n-1} + \sum_{n=1}^{m} f_{n-1} g_n z_n - \sum_{n=1}^{m} f_{n-1} g_n z_{n-1}\right.$$

$$+\sum_{n=1}^{m} f_n g_{n-1} z_n - \sum_{n=1}^{m} f_n g_{n-1} z_{n-1} + \sum_{n=1}^{m} f_{n-1} g_{n-1} z_n \Big\}.$$

$$\int_B f:g\ \delta z = \frac{1}{4}\Big\{\sum_{n=1}^{m} -f_n g_n z_{n-1} + \sum_{n=0}^{m-1} f_n g_{n+1} z_{n+1} - \sum_{n=0}^{m-1} f_n g_{n+1} z_n$$

$$+\sum_{n=1}^{m} f_n g_{n-1} z_n - \sum_{n=1}^{m} f_n g_{n-1} z_{n-1} + \sum_{n=0}^{m-1} f_n g_n z_{n+1}\Big\}$$

$$= \frac{1}{4}\Big\{\sum_{n=1}^{m-1} - f_n g_n z_{n-1} + \sum_{n=1}^{m-1} f_n g_{n+1} z_{n+1} - \sum_{n=1}^{m-1} f_n g_{n+1} z_n$$

$$+\sum_{n=1}^{m-1} f_n g_{n-1} z_n - \sum_{n=1}^{m-1} f_n g_{n-1} z_{n-1} + \sum_{n=1}^{m-1} f_n g_n z_{n+1}$$

$$- f_m g_m z_{m-1} + f_m g_{m+1} z_{m+1} - f_m g_{m+1} z_m$$

$$+ f_m g_{m-1} z_m - f_m g_{m-1} z_{m-1} + f_m g_m z_{m+1}\Big\}$$

$$= \frac{1}{4}\Big\{\sum_{n=1}^{m-1} f_n(-g_n z_{n-1} + g_{n+1} z_{n+1} - g_{n+1} z_n$$

$$+ g_{n-1} z_n - g_{n-1} z_{n-1} + g_n z_{n+1}).$$

$$+ f_m(-g_m z_{m-1} + g_{m+1} z_{m+1} - g_{m+1} z_m$$

$$+ g_{m-1} z_m - g_{m-1} z_{m-1} + g_m z_{m+1})\Big\}.$$

$$\int_B f{:}g\ \delta z = \frac{1}{4}\Big\{\sum_{n=1}^{m-1}\Big\{f_n(g_{n-1}[z_n - z_{n-1}] + g_n\ [z_{n+1} - z_{n-1}]$$

$$+ g_{n+1}\ [z_{n+1} - z_n])\Big\} + f_m\ \Big\{g_{m-1}(z_m - z_{m-1})$$

$$+ g_m(z_{m+1} - z_{m-1}) + g_{m+1}(z_{m+1} - z_m)\Big\}$$

$$(23)\qquad = \frac{1}{4}\sum f_n[g_{n-1}(z_n - z_{n-1}) + g_n(z_{n+1} - z_{n-1})$$

$$+ g_{n+1}\ (z_{n+1} - z_n)].$$

Similarly,

$$(24)\qquad \int_B f\ {:}g'\ \delta z = -\sum_{n=1}^{m} f_n(g_{n+1} - g_{n-1})/2,$$

and

$$(25)\qquad \int_B f'{:}g'\ \delta z = -\sum_{n=1}^{m} f_n\ \left[\ \frac{g_{n+1} - g_n}{z_{n+1} - z_n} - \frac{g_n - g_{n-1}}{z_n - z_{n-1}}\ \right].$$

As a matter of notation we take $z_{m+1} = z_1$.

As a simplifying notation let

$$2\bar{f}_0 = f_0 + f_1,\ 2\bar{f}_1 = f_1 + f_2,\ 2\bar{f}_2 = f_2 + f_3,\ 2\bar{f}_3 = f_3 + f_0.$$

<u>LEMMA</u> 2.1. $(1 - i)\, Lf = 2L\bar{f}$.

PROOF:

$$\begin{aligned} (1 - i)Lf &= (1 - i)(f_0 + if_1 - f_2 - if_3) \\ &= f_0 + if_1 - f_2 - if_3 - if_0 + f_1 \\ &\quad + if_2 - f_3 \\ &= (f_0 + f_1) + i(f_1 + f_2) - (f_2 + f_3) \\ &\quad - i(f_3 + f_0) \\ &= 2\bar{f}_0 + 2i\bar{f}_1 - 2\bar{f}_2 - 2i\bar{f}_3 \\ &= 2L\bar{f}. \end{aligned}$$

Q.E.D.

Similarly,

(26) $$Sf = S\bar{f} \text{ and } T\bar{f} = 0.$$

<u>LEMMA</u> 2.2. <u>If $\bar{g}$ is defined in the same manner then</u>

(27) $$\int f{:}g\ \delta z = L(\overline{fg}).$$

PROOF:

$$\begin{aligned} L(\overline{fg}) &= (\overline{fg})_0 + i(\overline{fg})_1 - (\overline{fg})_2 - i(\overline{fg})_3 \\ &= \overline{f_0g_0} + i\overline{f_1g_1} - \overline{f_2g_2} - i\overline{f_3g_3} \\ &= \bar{f}_0\bar{g}_0 + i\bar{f}_1\bar{g}_1 - \bar{f}_2\bar{g}_2 - i\bar{f}_3\bar{g}_3 \end{aligned}$$

$$L(fg) = \frac{(f_0 + f_1)}{2}\frac{(g_0 + g_1)}{2} - \frac{(f_2 + f_3)}{2}\frac{(g_2 + g_3)}{2}$$

$$+ i[\frac{(f_1 + f_2)}{2}\frac{(g_1 + g_2)}{2}$$

$$- \frac{(f_3 + f_0)}{2}\frac{(g_3 - g_0)}{2}].$$

While,

$$\int f{:}g\ \delta z = \frac{1}{4}[(f_1 + f_0)(g_1 + g_0)(1 - 0)$$

$$+ (f_2 - f_1)(g_2 + g_1)(1 + i - 1)$$

$$+ (f_3 + f_2)(g_3 + g_2)(i - 1 - i)$$

$$+ (f_0 + f_3)(g_0 + g_3)(0 - i)].$$

Q.E.D.

<u>LEMMA</u> 2.3. $4L(fg) = Mf + Tg + MgTf + SfLg + SgLf.$

PROOF: $Lf^2 = f_0^2 - f_2^2 + i(f_1^2 - f_3^2)$

$$= f_0^2 - f_2^2 + (f_1 + f_3)(f_2 - f_0) + (f_1 + f_3)Lf$$

(28) $$= (f_0 - f_2)\ Tf + (f_1 + f_3)Lf.$$

Similarly,

$$Lf^2 = i(f_1^2 - f_3^2) + i(f_3 - f_1)(f_0 + f_2)$$

$$+ (f_0 + f_2)\ Lf$$

(29) $$Lf^2 = i(f_3 - f_1)\, Tf + (f_0 + f_2)Lf.$$

Adding (28) and (29) yields

(30) $$2Lf^2 = MfTf + SfLf.$$

Note that $4fg = (f + g)^2 - (f - g)^2$.

Thus (30) gives the desired identity.

Q.E.D.

<u>THEOREM</u> 2.1. <u>If</u> B <u>is the boundary of a unit square</u> R <u>and if the contour is described in the counter clockwise sense, then</u>

(31) $$\int_B f : g\, \delta z = \frac{(1 - i)}{8} \sum_R (SgLf + SfLg),$$

(32) $$\int_B f : g'\delta z = \frac{i}{4} \sum_R (MgLf - MfLg),$$

<u>and</u>

(33) $$\int_B f':g'\delta z = \frac{(1 + i)}{2} \sum_R (TgLf + TfLg).$$

PROOF: Lemma 2.3, Lemma 2.1, and (29) imply that

(34) $$8L\overline{fg} = 2M\overline{f}T\overline{g} + 2M\overline{g}T\overline{f} + 2S\overline{f}L\overline{g} + 2S\overline{g}L\overline{f}$$

$$= Sf(1 - i)\, Lg + Sg(1 - i)\, Lf$$

(35) $$= (1 - i)\, [SfLg + SgLf]\, .$$

Combining (35) and (27) we obtain,

(36) $$\int f:g\ \delta z = \frac{(1-i)}{8}\ (SgLf + SfLg).$$

This proves (31) for a unit square.

Now consider (33). Let F be the dual of f and let G be the dual of g. Then relation (31) holds for F and G. We take the complex conjugate of this relation. Note that $(SG)^* = \pm Tg$ and $(LF)^* = \pm Lf$. Thus $(SGLF)^* = TgLf$. Also $(F_n + F_{n-1})^* = \pm (f_n - f_{n-1})$ and $(z_n - z_{n-1})^* = (z_n - z_{n-1})^{-1}$. Thus

$$(4\int_a^b F:G\ \delta z)^* = \int_a^b f':g'\ \delta z\ .$$

It is now clear that the complex conjugate of (31) for G and F yields (33) for f and g.

$$\begin{aligned} 2\int_B f:g'\delta z &= f_1(g_2 - g_o) + f_2(g_3 - g_1) \\ &\quad + f_3(g_o - g_2) + f_o(g_1 - g_3) \\ &= (f_o - f_2)(g_1 - g_3) - (f_1 - f_3)(g_o - g_2). \end{aligned}$$

But $Lf + Mf = 2(f_o - f_2)$ and $Lf - Mf = 2i(f_1 - f_3)$.

$$\text{So } 8i \int f{:}g'\delta z = (Lf + Mf)(Lg - Mg)$$
$$-(Lg + Mg)(Lf - Mf)$$
$$= 2MfLg - 2MgLf,$$

or

$$\int f{:}g'\delta z = \frac{i}{4}[-MfLg + MgLf]$$

This verifies (32) for a unit square.

Q.E.D.

<u>COROLLARY</u> 2.1.1. <u>If</u> B <u>is the boundary of a simple region</u> R <u>and the contour is described in the counter clockwise sense, then</u> (31), (32), <u>and</u> (33) <u>still hold, where</u> $\sum_R$ <u>again denotes a summation over the unit squares of</u> R.

PROOF: This follows from the proof of Theorem 2.1 by juxtaposition. The line integral on edges common to two squares will cancel leaving only the line integral on B.

Q.E.D.

<u>THEOREM</u> 2.2. <u>If</u> B <u>is a closed circuit then</u>

$$\int_B f{:}g'\delta z = -\int_B g{:}f'\delta z.$$

PROOF: Directly from definition (21) it is seen that

$$\int_a^b f{:}g'\,\delta z + \int_a^b g{:}f'\delta z = \sum_{n=1}^{m} (f_n g_n - f_{n-1} g_{n-1}).$$

Thus

(37) $$\int_a^b (f:g' + g:f')\delta z = f(a)g(a) - f(b)g(b).$$

In particular for a closed circuit,

$$\int_B f:g'\delta z = -\int_B g:f'\delta z.$$

Q.E.D.

Suppose that the path of integration is in a simple region where g is analytic. Then the derivative h of g exists. If z_{n-1} and z_n are two consecutive points on the path of integration

$$(h_n + h_{n-1})(z_n - z_{n-1}) = 2(g_n - g_{n-1}).$$

Comparing (20) and (21) gives.

(39) $$\int_a^b f:g'\delta z = \int_a^b f:h\ \delta z.$$

Comparing (21) and (22) gives

(40) $$\int_a^b f':g'\delta z = \int_a^b f':h\ \delta z.$$

Let j be the second derivative of g. Considering a closed path of integration it follows that

(41) $$\int_B f':g'\delta z = \int_B f':h\ \delta z \qquad \text{(by (40))}$$

$$= -\int_B f:h'\delta z \qquad \text{(by (38))}$$

$$\int_B f':g'\delta z = -\int_B f:j\ \delta z \qquad \text{(by (39))}.$$

Whether or not g is analytic, it is convenient to introduce the following notation.

(41a) $$\int_B f':g'\delta z = -\int_B f:g''\delta z$$

Analogs of the Cauchy integral formula will now be obtained. To this end a lattice function q is supposed given such that

(42) $$Lq(z) = 0\ ,\ z \neq 0 \text{ and } Lq(0) = 1.$$

If h is a function which is analytic everywhere, then $q' = q + h$ also satisfies the condition (42). Let f be analytic in a simple region R which includes the unit square at $z = 0$. Then (31), (32), and (33) yield

(43) $$\int_B f:q\ \delta z = (1 - i)Sf/8, \qquad \text{(for } z = 0\text{)}$$

(44) $$\int_B f:q'\delta z = -iMf/4\ ,$$

(45) $$\int_B f:q''\delta z = -(1 + i)Tf/2.$$

It is seen from the definitions of L, M, T, and S that

(46) $$\begin{aligned} 4f_o &= (S + T + M + L)f\ , \\ 4f_1 &= (S - T + iM - iL)f\ , \end{aligned}$$

$$4f_2 = (S + T - M - L)f\,, \quad \text{and}$$

(46)

$$4f_3 = (S - T - iM + iL)f.$$

It follows from relations (43) to (46) that

$$f_o = \int_B f\colon \frac{1}{4}\,[(1 + i)q + iq' - (1 - i)q'']\delta z\ ,$$

$$f_1 = \int_B f\colon \frac{1}{4}\,[(1 + i)q - q' + (1 - i)q'']\delta z\ ,$$

(47)

$$f_2 = \int_B f\colon \frac{1}{4}\,[(1 + i)q - iq' - (1 - i)q'']\delta z\ ,$$

and

$$f_3 = \int_B f\colon \frac{1}{4}\,[1 + i)q + q' + (1 - i)q'']\delta z\ .$$

Here $z_o = 0$, $z_1 = 1$, $z_2 = 1 + i$, and $z_3 = i$. The "factoring" of f in relations (47) is a matter of symbolism.

Relations (47) are analogs of the Cauchy integral formula. It is of interest to obtain a more symmetric Cauchy type formula. To this end let

$$S' = I + X^{-1} + Y^{-1}X^{-1} + Y^{-1}$$

and

$$T' = I - X^{-1} + Y^{-1}X^{-1} - Y^{-1}\ .$$

Then

$$S'S = (I + X^{-1} + Y^{-1}X^{-1} + Y^{-1})(I + X + YX + Y)$$

$$= I + X + YX + Y + X^{-1} + I + Y + X^{-1}Y$$

$$+ Y^{-1}X^{-1} + I + X^{-1} + Y^{-1} + XY^{-1} + X + I$$

$$S'S = 4I + XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y + 2(X + X^{-1} + Y + Y^{-1}),$$

and, similarly,

$$T'T = 4I + XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y - 2(X + X^{-1} + Y + Y^{-1}).$$

Thus

$$\begin{aligned} S'S + T'T &= 4I + XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y \\ &\quad + 2(X + X^{-1} + Y + Y^{-1}) \\ &\quad + 4I + XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y \\ &\quad - 2(X + X^{-1} + Y + Y^{-1}) \\ &= 8I + 2(XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y) \\ &= 8I + 2(D + 4I) \end{aligned}$$

$$(48) \qquad = 16I + 2D.$$

Let

$$(49) \qquad s(z) = \frac{(1 + i)}{4} Sq(z) \text{ and } t(z) = \frac{(1 - i)}{16} Tq(z).$$

Let z_o be an interior point of R then by (43)

$$\begin{aligned} &4(1 + i) \int_B f(z):g(z - z_o)\, \delta z \\ &= 4 \frac{(1 - i)(1 + i)}{8} Sf(z_o) \\ &= Sf(z_o), \end{aligned}$$

Operating with S' gives

$$S'Sf(z_o) = S'[4(1+i)\int_B f(z):q(z-z_o)\delta z]$$

$$= 4(1+i)\ [\int_B f(z):q(z-z_o)\delta z$$

$$+ \int_B f(z):q(z-z_o+1)\delta z$$

$$+ \int_B f(z):q(z-z_o+1+i)\delta z$$

$$+ \int_B f(z):q(z-z_o+i)\delta z]$$

$$= 4(1+i)\ [\int_B f(z):(q(z-z_o) + q(z-z_o+1)$$

$$+ q(z-z_o+1+i) + q(z-z_o+i))\delta z]$$

$$= 4(1+i)\ [\int_B f(z):Sq(z-z_o)\delta z]$$

$$= 4(1+i)\ [\int_B f(z): \frac{4s(z-z_o)}{(1+i)}\ \delta z]$$

$$= 16\int_B f(z):s(z-z_o)\delta z\ .$$

Similarly, (45) gives

$$T'Tf(z_o) = 16i \int_B f(z):t''(z - z_o)\delta z.$$

Relation (48) gives

$$16 \int_B f(z):s(z - z_o)\delta z + 16\underline{i} \int_B f(z):t''(z - z_o)\delta z$$

$$= 16f(z_o) + zDf(z_o).$$

Since Df = 0 at an interior point z_o,

$$(50) \qquad \int_B f(z):[s(z - z_o) + it''(z - z_o)]\delta z = f(z_o).$$

This is another analog of the Cauchy formula.

3. The Product of Analytic Functions[1]

This section is concerned with operators which correspond to multiplication of a continuous analytic function by z. The formulas prove to be somewhat more complicated than those in a similar theory developed by Isaacs for type (a) analytic functions.

In the formula

$$4Lfg = MfTg + MgTf + SfLg + SgLf$$

the function f is to be taken as z or z^*. Note that $Lz = 0$, $Tz = 0$, $Mz = -4\delta$, and $Sz = 4z + 4\delta$. Here δ denotes $(1 + i)/2$. Thus

$$Lzg = \frac{1}{4}\,[MzTg + MgTz + SzLg + SgLz]$$

$$= \frac{1}{4}\,[-4\delta Tg + Mg \cdot 0 + (4z + 4\delta)Lg + Sg \cdot 0]$$

$$(51) \qquad = -\delta Tg + (z + \delta)\,Lg.$$

Also $Mz^* = 0$, $Tz^* = 0$, $Lz^* = -4\delta^*$; and $Sz^* = 4z^* + 4\delta^*$. Thus

$$(52) \qquad Lz^*g = -\,\delta^*Sg + (z + \delta)^*Lg$$

In (51) take $g = Sf$, in (52) take $g = iTf$, and subtract. This yields

$$LzSf - Lz^*iTf = -\delta TSf + (z + \delta)LSf + \delta^*SiTf - (z + \delta)^*LiTf\,.$$

$$(53) \qquad L(zS - iz^*T)f = [(z + \delta)S - i(z + \delta)^*T]Lf,$$

[1]The definitions and results appearing in this chapter may be found in [1].

since

$$[-\frac{(1+i)}{2}TS + \frac{(1-i)}{2}SiT]f = [-\frac{(1+i)}{2} + \frac{i(1-i)}{2}]\,TSf$$

$$= [\frac{-1-i+i+1}{2}]\,TSf$$

$$= 0.$$

<u>DEFINITION</u> 3.1. Let the <u>operator</u> <u>Z</u> be defined by

$$4Z = zS - iz^*T. \tag{54}$$

<u>THEOREM</u> 3.1. <u>If</u> f <u>is</u> <u>descrete</u> <u>analytic</u> <u>then</u> Zf <u>is</u> <u>discrete</u> <u>analytic</u>.

PROOF: From (53) we see that

$$Lf = 0 \text{ implies } LZf = 0.$$

Q.E.D.

It is seen that the operator Z has a correspondence with multiplication by z in the classical continuous theory.

It is of interest to express Z in other forms. We have

$$4Zf_o = (zS - iz^*T)f_o$$

$$= zSf_o - iz^*Tf_o$$

$$= z[f_o + f_1 + f_2 + f_3] - iz^*[f_o - f_1 + f_2 - f_3]$$

$$= (z - iz^*)(f_o + f_2) + (z + iz^*)(f_1 + f_3).$$

Now

$$z + iz^* = x + iy + i\ (x - iy)$$

$$= x + y + i\ (y + x)$$

$$= (1 + i)(x + y)$$

$$= 2\delta(x + y),$$

and

$$z - iz^* = 2\delta^*(x - y).$$

Thus

$$4Zf_o = 2\delta^*(x - y)(f_o + f_2) + 2\delta(x + y)(f_1 + f_3)$$

$$4\delta Zf_o = 2\delta\delta^*(x - y)(f_o + f_2) + 2\delta^2(x + y)(f_1 + f_3)$$

$$= 2\ \frac{(1 + i)}{2}\ \frac{(1 - i)}{2}\ (x - y)(f_o + f_2)$$

$$+ 2\ \frac{(1 + i)^2}{4}\ (x - y)(f_1 + f_3)$$

$$= 2\ \frac{(1 + 1)}{4}\ (x - y)(f_o + f_2) + 2\ \frac{(2i)}{4}\ (x - y)(f_1 + f_3)$$

$$= (x - y)(f_o + f_2) + i(x - y)(f_1 + f_3). \qquad (55)$$

Suppose that $f = L'w$. Employing the spiral coordinate system gives:

$$f_o = w_o - w_6 + iw_7 - iw_5\ ,$$

$$f_1 = w_1 - w_7 + iw_8 - iw_o\ ,$$

$$f_2 = w_2 - w_o + iw_1 - iw_3\ , \qquad \text{and}$$

$$f_3 = w_3 - w_5 + iw_o - iw_4\ .$$

Substituting this in (55) yields

(56) $$4\delta Z f_o = (x - y)(w_2 - w_6) + (x + y)(w_4 - w_8)$$
$$+ 2i[x(w_1 - w_5) + y(w_3 - w_7)].$$

Let w be a real harmonic function; then f is analytic. Thus it follows from (56) that

$$U = (x - y)(XY - X^{-1}Y^{-1})w + (x + y)(X^{-1}Y - XY^{-1})w$$

(57) and

$$V = 2x(X - X^{-1})w + 2y(Y - Y^{-1})w$$

are a pair of harmonic conjugates.

DEFINITION 3.2. Let an operator Q be defined by

$$4Q = ZS'$$

where $S' = I + X^{-1} + Y^{-1} + X^{-1}Y^{-1}$.

THEOREM 3.2. If f is analytic then Qf is analytic.

PROOF:

$$16Q = 4ZS'$$
$$= 4\,\frac{(zS - iz^{*}T)}{4}\,S'$$
$$= zSS' - iz^{*}TS'$$

(58) $$= zA - iz^{*}B ,$$

where $A = SS'$ and $B = TS'$.

In terms of the spiral coordinate notation we have,

$$Af_o = SS'f_o$$

$$= S(f_o + f_5 + f_7 + f_6)$$

$$= f_o + f_1 + f_2 + f_3 + f_5 + f_o + f_3 + f_4$$

$$+ f_7 + f_8 + f_1 + f_o + f_6 + f_7 + f_o + f_5$$

(59) $$= 4f_o + 2f_1 + 2f_3 + 2f_5 + 2f_7 + 2f_2 + f_4 + f_6 + f_8 ,$$

and, similarly,

(60) $$Bf_o = f_2 - f_4 + f_6 - f_8 .$$

It follows from (53) that

$$16LQf = [(z + \delta)A + (z + \delta)^* B]Lf.$$

Thus $Lf = 0$ implies $LQf = 0$, and Qf is analytic if f is analytic.

Q.E.D.

The operators Z and Q are similar; however, Q has greater symmetry to the point of application.

In order to study the properties of Q it is useful to introduce an analog of the function e^{zt}.

DEFINITION 3.3. Let $\underline{e}(z,t)$ be defined by

$$e(z,t) = \left(\frac{2 + t}{2 - t} \right)^x \left(\frac{2 + it}{2 - it} \right)^y$$

where $z = x + iy$.

This function was introduced by Jacqueline Ferrand. A similar function was employed by Rufus Isaacs to study multiplication.

<u>THEOREM</u> 3.3. $e(z,t)$ <u>is discrete analytic</u>.

PROOF:
$$\begin{aligned}
Le(z,t) &= e(z,t) + ie(z+1,t) \\
&\quad - e(z+i+1,t) - ie(z+i,t) \\
&= \left(\frac{2+t}{2-t}\right)^x \left(\frac{2+it}{2-it}\right)^y \\
&\quad + i\left(\frac{2+t}{2-t}\right)^{x+1} \left(\frac{2+it}{2-it}\right)^y \\
&\quad - \left(\frac{2+t}{2-t}\right)^{x+1} \left(\frac{2+it}{2-it}\right)^{y+1} \\
&\quad - i\left(\frac{2+t}{2-t}\right)^x \left(\frac{2+it}{2-it}\right)^{y+1} \\
&= \left(\frac{2+t}{2-t}\right)^x \left(\frac{2+it}{2-it}\right)^y \Big[1 + i\left(\frac{2+t}{2-t}\right) \\
&\quad - \left(\frac{2+t}{2-t}\right)\left(\frac{2+it}{2-it}\right) - i\left(\frac{2+it}{2-it}\right)\Big] \\
&= \frac{(2+t)^x}{(2-t)^{x+1}} \frac{(2+it)^y}{(2-it)^{y+1}} \big[(2-t)(2-it) \\
&\quad + i(2+t)(2-it) - (2+t)(2+it) \\
&\quad - i(2+it)(2-t)\big] \\
&= \frac{(2+t)^x}{(2-t)^{x+1}} \frac{(2+it)^y}{(2-it)^{y+1}} [4 - 2t - 2it
\end{aligned}$$

$$+ it^2 + 4i + 2it + 2t + t^2 - 4 - 2t$$

$$- 2it - it^2 - 4i + 2t + 2it - t^2]$$

$Le(z,t) = 0.$

Q.E.D.

<u>THEOREM</u> 3.4.

$$1 + t\int_0^z e(z,t)\delta z = e(z,t)$$

PROOF: We will show that the theorem is true for any one step. Since the only possibilities are ± 1 and $\pm i$, we will investigate +1 and +i ; the proof for -1 and -i are similar.

$$\int_{z-1}^{z} e(z,t)\delta t = \frac{1}{2}[e(z,t) + e(z-1,t)] \quad (1)$$

$$= \frac{1}{2} e(z,t)\ [1 + (\frac{2-t}{2+t})]$$

$$= e(z,t)\ [\frac{2 + t + 2 - t}{2(2+t)}]$$

$$= e(z,t)[\frac{2}{2+t}]$$

$$= e(z,t)[\frac{2t}{2+t}]\ \frac{1}{t}$$

$$= e(z,t)[\frac{2 + t - 2 + t}{2+t}]\ \frac{1}{t}$$

$$\int_{z-1}^{z} e(z,t)\delta t = e(z,t)[1 - \frac{2-t}{2+t}]\frac{1}{t}$$

$$= \frac{e(z,t) - e(z-1,t)}{t}, \quad \text{and}$$

$$\int_{z-i}^{z} e(z,t)\delta t = \frac{1}{2}[e(z,t) + e(z-i,t)](i)$$

$$= \frac{i}{2} e(z,t)[1 + (\frac{2-it}{2+it})]$$

$$= ie(z,t)[\frac{2+it+2-it}{2(2+it)}]$$

$$= e(z,t)[\frac{2i}{2+it}]$$

$$= e(z,t)\ [\frac{2it}{2+it}]\frac{1}{t}$$

$$= e(z,t)[\frac{2+it-2+it}{2+it}]\frac{1}{t}$$

$$= e(z,t)[1 - (\frac{2-it}{2+it})]\frac{1}{t}$$

$$= \frac{e(z,t) - e(z-i,t)}{t}.$$

Now assume z is in the first quadrant, and $z = x + iy$.

$$\int_0^z e(z,t)\delta t = \int_0^1 e(z,t)\delta t + \int_1^2 e(z,t)\delta t + \ldots$$

$$+\int_{x-1}^{x} e(z,t)\delta t + \int_{x}^{x+i} e(z,t)\delta t$$

$$+\int_{x+i}^{x+2i} e(z,t)\delta t + \ldots + \int_{x+(y-1)i}^{x+yi} e(z,t)\delta t.$$

$$\int_0^z e(z,t)\delta t = \frac{e(1,t) - e(0,t)}{t} + \frac{e(2,t) - e(1,t)}{t}$$

$$+\ldots+ \frac{e(x,t) - e(x-1,t)}{t}$$

$$+ \frac{e(x+i,t) - e(x,t)}{t}$$

$$+ \frac{e(x+2i,t) - e(x+i,t)}{t} + \ldots$$

$$+ \frac{e(x+yi,t) - e(x+(y-1)i,t)}{t}$$

$$= \frac{1}{t}\,[e(x+yi,t) - e(x+(y-1)i,t)$$

$$+ e(x+(y-1)i,t) - \ldots - e(x,t)$$

$$+ e(x,t) - e(x-1,t) + \ldots + e(2,t)$$

$$- e(1,t) + e(1,t) - e(0,t)]$$

$$\int_0^z e(z,t)\delta t = \frac{1}{t}\,[e(x + yi,t) - e(0,t)]$$

$$= \frac{1}{t}\,[e(z,t) - 1]\ .$$

Q.E.D.

This relation brings out an analogy with e^{zt}.

If $|t| < 2$ a power series expansion in t is valid. Thus

(61) $$e(z,t) = \sum_{n=0}^{\infty} \frac{z^{(n)}t^n}{n!}\ .$$

By applying the binomial theorem to (61) it would be possible to obtain a formula for the coefficients, $z^{(n)}$. For the present purpose it is sufficient to observe that they are polynomials in x and y. In fact it follows from Theorem 3.4 that

$$z^{(n+1)} = (n + 1)\int_0^z z^{(n)}\ \delta z$$

and

$$z^{(0)} = 1.$$

<u>THEOREM</u> 3.5

$$Qz^{(n)} = z^{(n+1)}$$

PROOF: A straightforward calculation using Definition 3.2 and Definition 3.3 gives,

$$Qe(z,t) = e(z,t)(16z + 4t^2z^*)(16 - t^4)^{-1}.$$

It is readily verified that the same expression on the right is obtained by differentiating $e(z,t)$ with respect to t. Thus

$$Qe(z,t) = \partial e(z,t)/\partial t. \tag{62}$$

Substituting (61) in (62) yields

$$\sum_{n=0}^{\infty} t^n Qz^{(n)}/n! = \sum_{n=0}^{\infty} t^n z^{(n+1)}/n!$$

Equating the coefficients to t^n yields

$$Qz^{(n)} = z^{(n+1)}.$$

Q.E.D.

4. A Convolution Product[1]

DEFINITION 4.1. The convolution product of the functions f and g, $f*g$, is defined as

$$f*g = \int_0^z f(z - t):g(t)\delta t. \tag{63}$$

In this section we shall assume f and g are defined in a common simple region R which has a boundary consisting of a simple closed chain. Such a region will be termed simply connected.

Equation (63) requires that not only the chain $0 = z_o, z_1, z_2, \ldots, z_m = z$ over which we take the double dot line integral lies in R, but also the chain $z - z_o$, $z - z_1, \ldots,$ $z - z_m$ lies in R. Thus we make the following definition.

DEFINITION 4.2. A simply connected region R with the additional property that for every point z in R there exists at least one chain $0 = z_o, z_1, \ldots, z_m = z$ in R such that $z - z_o$, $z - z_1, \ldots, z - z_m$ is also in R, will be called a convolution region.

DEFINITION 4.3. The chain $z - z_o$, $z - z_1, \ldots, z - z_m$ is called the counter chain of $z_o, z_1, \ldots, z_m$.

---

[1] The definitions and results appearing in this chapter may be found in [2].

Suppose every point z of a region can be reached from the fixed point z_o by a chain which has a center of symmetry. Then this region is a convolution region relative to z_o, because the counter chain is simply the original chain traversed in reverse order. To see this note that by symmetry $z_i + z_{m-i} = z_o + z_m = z$. Thus $z_i = z - z_{m-i}$ and the right side is the counter chain in reverse order.

It is instructive to consider the corresponding question of convolution regions for the continuous case. It is seen that every region which is star-like relative to z_o is a convolution region because the chain maybe taken to be the straight line going from z_o to z. The counter chain is then the line from z to z_o.

We now are able to state the following theorem:

<u>THEOREM</u> 4.1. <u>If</u> f(z) <u>and</u> g(z) <u>are</u> <u>discrete</u> <u>analytic</u> <u>in</u> <u>a</u> <u>convolution</u> <u>region</u> R, <u>then</u>

$$(64) \qquad f*g = \int_0^z f(z - t):g(t)\delta t = \int_0^z g(z - t):f(t)\delta t$$

<u>is</u> <u>discrete</u> <u>analytic</u> <u>in</u> R.

PROOF:

$$L(f*g) = \int_0^z f(z - t):g(t)\delta t$$

$$+ i\int_0^{z+1} f(z + 1 - t):g(t)\delta t$$

$$- \int_0^{z+1+i} f(z + 1 + i - t):g(t)\delta t$$

$$- i \int_0^{z+i} f(z + i - t):g(t)\delta t$$

$$L(f*g) = \int_0^{z} f(z - t):g(t)\delta t$$

$$+ i[\int_0^{z} f(z + 1 - t):g(t)\delta t$$

$$+ \int_z^{z+1} f(z + 1 - t):g(t)\delta t] -$$

$$- [\int_0^{z} f(z + 1 + i - t):g(t)\delta t$$

$$+ \int_z^{z+1+i} f(z + 1 + i - t):g(t)\delta t]$$

$$- i[\int_0^{z} f(z + i - t):g(t)\delta t$$

$$+ \int_z^{z+i} f(z + i - t):g(t)\delta t]$$

$$= \int_0^{z} [f(z - t) + if(z + 1 - t)$$

$$- f(z + 1 + i - t)$$

$$- if(z + i - t)]:g(t)\delta t$$

$$+ i \int_{z}^{z+1} f(z + 1 - t):g(t)\delta t$$

$$- \int_{z}^{z+1+i} f(z + 1 + i - t):g(t)\delta t$$

$$- i \int_{z}^{z+i} f(z + i - t):g(t)\delta t$$

(65) $$L(f*g) = \int_{\Theta}^{z} L[f(z - t)]:g(t)\delta t$$

$$+ \int_{z}^{z+1} f(z + 1 - t):g(t)\delta t$$

$$- \int_{z}^{z+1+i} f(z + 1 + i - t):g(t)\delta t$$

$$- i \int_{z}^{z+i} f(z + i - t):g(t)\delta t.$$

Since f(z) is discrete analytic at z, the first integral in (65) is zero. Thus

$$L(f*g) = i\int_{z}^{z+1} f(z+1-t):g(t)\delta t$$

$$- \int_{z}^{z+1+i} f(z+1+i-t):g(t)\delta t$$

$$- i\int_{z}^{z+i} f(z+i-t):g(t)\delta t.$$

Using the definition of the double dot integral we obtain,

$$4L(f*g) = i[f(z_1 - z_1)\ 2\ f(z_1 - z_o)][g(z_1)$$

$$+ g(z_o)](z_1 - z_o)$$

$$- [f(z_2 - z_2) + f(z_2 - z_1)][g(z_2)$$

$$+ g(z_1)](z_2 - z_1)$$

$$- [f(z_2 - z_1) + f(z_2 - z_o)][g(z_1)$$

$$+ g(z_o)](z_1 - z_o)$$

$$- i[f(z_3 - z_3) + f(z_3 - z_o)][g(z_3)$$

$$+ g(z_o)](z_3 - z_o)$$

$$= i[f(0) + f(1)][g(z_1) + g(z_o)](1)$$

$$- [f(0) + f(i)][g(z_2) + g(z_1)](i)$$

$$- [f(i) + f(i+1)][g(z_1) + g(z_o)](1)$$

$$- i\,[f(0) + f(i)][g(z_3) + g(z_o)](i).$$

Using the fact that $f(i + 1) = f(0) + if(1) - if(i)$ we obtain,

$$L(f*g) = \frac{i}{4}\,[f(0) + f(i)]Lg(z).$$

Thus if $g(z)$ is discrete analytic, $Lg(z) = 0$, which implies $L(f*g) = 0$, which implies $f*g$ is discrete analytic.

To prove relation (64) let $s = z - t$, then by the definition of the double dot integral

$$\int_0^z f(z - t):g(t)\delta t = -\int_z^0 f(s):g(z - s)\delta s$$

$$= \int_0^z g(z - r):f(r)\delta r.$$

Here the summation on s is taken over the counter chain, and the summation on r runs over the counter chain in the reverse order. Thus (64) follows because the double dot integral is independent of path.

Q.E.D.

From theorem 4.1 we see that the convolution product is commutative and preserves discrete analyticity in convolution regions. The distributivity follows easily from the definition of the double dot integral. The next theorem concerns the associativity of the convolution product. The details in proving associativity for convolution regions are not completely understood, for this reason the theorem is stated and proved for simply connected rectangular regions which include the origin.

THEOREM 4.2. *If* f, g *and* h *are discrete analytic in a rectangular region* R *containing the origin, then*

$$f * (g * h) = (f * g) * h.$$

PROOF: Since R is rectangular and simply connected, the functions f, g and h may be continued in a non-unique way as discrete analytic functions in the entire discrete complex plane. The continuation may be carried out by defining arbitrarily the values of f, g and h for the portions of the x and y axes not included in R. Using the defining relation

$$Lf(z_o) = f(z_o) + if(z_o + 1) - f(z_o + 1 + i) - if(z_o + i)$$

and the values of f, g and h in R and on the x and y axes we may continue f, g and h to the entire lattice as discrete analytic functions. Now consider

$$f * (g * h) = \int_0^w f(w - z) : \int_0^z g(z - t) : h(t) \delta t \delta z.$$

Let $(z_o, z_1, \ldots, z_m)$ be a chain in R joining 0 and w such that the counter chain $(w - z_o, w - z_1, \ldots, w - z_m)$ is in R. Considering the line integral over this chain, we define

$$\mu(z_i, z_j) = g(z_i - z_j) \quad \text{if } i \geq j \quad \text{and}$$

$$\mu(z_i, z_j) = (-1)^{j-i} g(0) \quad \text{if } i < j\,.$$

From the definition of the double dot integral we have,

$$f * (g * h) = \int_0^w f(w - z) : \int_0^w \mu(z,t) : h(t) \delta t \delta z,$$

over the given chain. Interchanging the order of discrete integration gives

$$f * (g * h) = \int_0^w h(t) : \int_0^w f(w - z) : \mu(z,t) \delta z \delta t.$$

Using the definition of $\mu(z,t)$, this becomes

$$f * (g * h) = \int_0^w h(t) : \int_t^w f(w - z) : g(z - t) \delta z \delta t.$$

Let $\tau = w - z$; then $z - t = w - \tau - t$. Then

$$f * (g * h) = \int_0^w h(t) : \int_0^{w-t} f(\tau) : g(w - \tau - t) \delta \tau \delta t.$$

Let $\zeta = w - t$; then $t = w - \zeta$ which gives

$$f * (g * h) = \int_0^w h(w - \zeta) : \int_0^{\zeta} f(\tau) : g(\zeta - \tau) \delta \tau \delta \zeta.$$

Thus $f * (g * h) = h * (g * f) = (f * g) * h$.

Q.E.D.

It has now been shown that the convolution product satisfies the distributive, associative, and commutative laws as does ordinary multiplication. Moreover the convolution product preserves discrete analyticity, while ordinary multiplication does not.

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